Limit Behavior of Shortfall Risk in Incomplete Market with an Untradable Asset

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ABSTRACT. In this paper we study the shortfall risk convergence for American options (more generally for vanilla options). The financial market is provided with a non-traded asset, hence it is incomplete. The geometric Brownian motions are approximated by binomial trees.

1. Introduction

Though continuous trading is a part of the standard paradigm of modern finance, in practice, usually, portfolio revisions are done along a discrete–time greed. The links between discrete and continuous–time models have to be studied and some paradoxes appear. For example, the basic discrete approximation of a continuous asset price process may not lead to the convergence of the option price. In [7], the complete multi-assets model driven by geometric Brownian motions is approximated by binomial trees. In this context, the convergence of the shortfall risk for an American option is stated.

We consider a three assets model (S^0, S^1, S^2). The bank account S^0 is riskless. The risky assets (S^1, S^2) are driven by uncorrelated geometric Brownian motions. We assume that the asset S^2 is untradable. Then, the market is incomplete. Hedging an American option in this market will often lead to infinite initial endowment. It is then interesting to look at non perfect hedging and to compute the expected shortfall risk associated with the initial position.

We approximate the diffusions by piecewise constant processes. That is the markets are not active on intervals with length tending to zero. These models are in one-to-one correspondence with discrete-time models driven by binomial trees. Using this approximation, the expected shortfall risk for the vanilla options converges. This is the main result of the paper, see Theorem 3.1 below.

As in [7], we study the convergence of underlying processes (price processes, equivalent martingale densities, etc.) in a weak sense, mainly the convergence of laws in the Skorohod topology. The main technical novelty is the convergence of almost optimal strategies for the piecewise constant models to an admissible strategy for the continuous model, see Lemma 4.12 below. The extension of the results in [7] in the framework of incomplete markets is a direct consequence of the convergence of almost optimal strategies.

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The result is stated for markets with only two risky assets to alleviate notations. The generalization to models with more traded assets is barely difficult. The paper is organized as follows. In the next section, notations are detailed. In Section 3, we introduce the framework and Theorem 3.1 gives the main result of this paper. The technical Section 4 states several convergence results, and in particular the convergence of almost optimal strategies. Finally, in Sections 5 and 6, we use the results of Section 4 to prove the main result.

2. Notations

We use the following notations:

- $\mathbb{D}(\mathbb{R}^d)$ is the Skorohod space of the càdlàg functions $x : [0,T] \to \mathbb{R}^d$ while $\mathbb{C}(\mathbb{R}^d)$ denotes the space of continuous functions taking values in $\mathbb{R}^d$ with the uniform norm

$$||x||_T = \sup_{t \le T} |x_t|.$$

  For a survey of the Skorohod topology, weak convergence in the Skorohod space and corresponding notations we refer to [8].

- For a process H, we write in short

$$H \cdot W_t := \int_0^t H_u dW_u.$$

- For a semi-martingale L, the Doléans-Dade exponential Y (or stochastic exponential) is the solution of the stochastic equation

$$Y = 1 + Y_- \cdot L.$$

  The process Y is denoted by $\mathcal{E}(L)$. For more information about Doléans-Dade exponential, see Jacod and Shiryaev [8], Section I.4.f.

- For a sequence of non-negative real valued random variables $(\xi^n)_{n \in \mathbb{N}}$ dominated by a non-negative sequence $\{b^n\}_{n \in \mathbb{N}}$ with $b^n \le K n^{-a}$, we write

$$\xi^n \le O(n^{-a}).$$

  This notation allows to change the sequence $\{b^n\}$ and the constant K from line to line without further mention. Similarly, C may designate different constants which are independent of any variable. We use the notation C_m when the constant C_m depends on a parameter m. The constant C_m may also change from line to line.

3. Model and main result

Continuous-time model

We consider a three assets financial market. One asset is the numéraire (bank account) while the two others are risky. Their price processes are (independent) geometric Brownian motions $S = (S^1, S^2)$ defined on a filtered probability space $(\Omega, \mathcal{F}, \mathbf{F} = (\mathcal{F}_t), P)$:

$$dS_t^1/S_t^1 = \mu_1 dt + \sigma_1 dW_t^1, \tag{1}$$

$$dS_t^2/S_t^2 = \sigma_2 dW_t^2, \tag{2}$$

where $W = (W^1, W^2)$ is a standard 2-dimensional Brownian motion and the σ-algebra $\mathcal{F}_t = \sigma\{W_s, s \le t\}$. We assume S^2 is untradable and, therefore, the market is incomplete. Our aim is to price an American option with the pay-off function of the form

$$Y_t = F(t, S),$$

where $F : [0,T] \times \mathbb{D}(\mathbb{R}^2) \to \mathbb{R}_+$ is continuous (in the product of usual topology on $[0,T]$ and the Skorohod topology on $\mathbb{D}(\mathbb{R}^2)$), non-anticipating ($F(t,x) = F(t,y)$ if $x_s = y_s$ for $s \le t$), and satisfies the linear growth condition:

- there exists a constant $C \ge 0$,

$$\sup_{t \le T} F(t,x) \le C||x||_T, \quad \forall x \in \mathbb{D}(\mathbb{R}^2). \tag{3}$$

Note that this pay-off function characterizes a more general class of options, namely vanilla options.

A self-financing strategy is an adapted càdlàg process π such that the portfolio value is given by

$$V^{\pi,x} := x + \pi_- \cdot S^1,$$

where x is the initial capital; the strategy π is *admissible* if $V^{\pi,x} \ge 0$; the set of such strategies is denoted by $\mathcal{A}_x$.

Let $\mathcal{T}$ the set of all stopping times $\tau \le T$. We define *shortfall risk for an admissible strategy* π by

$$R(\pi, x) := \sup_{\tau \in \mathcal{T}} E(Y_\tau - V_\tau^{\pi,x})^+,$$

and the *shortfall risk for an initial capital* x:

$$R(x) := \inf_{\pi \in \mathcal{A}_x} R(\pi, x).$$

Approximating models

In this paragraph, we consider continuous-time models which approximate the financial market by means of piecewise constant price processes.

For any n, let a double indexed sequence of i.i.d. random variables $(\xi_k^i)_{k\le n}^{i=1,2}$ where ξ_k^i takes values in $\{-1,1\}$ and $P(\xi_k^i=1)=1/2$. Set

$$t_k^n = kT/n.$$

To alleviate notations, we write $t_k = t_k^n$ when there is no ambiguity. We set the filtration $\mathbf{F^n} = (\mathcal{F}_t^n)$ with $\mathcal{F}_t^n = \sigma\{(\xi_k^1, \xi_k^2),\ t_k^n \le t\}$ and a piecewise constant $\mathbf{F^n}$-adapted process $S^n = (S^{1n}, S^{2n})$,

$$S_t^{1n} = \prod_{m=1}^{k} \left(1 + \mu_1 \frac{T}{n} + \sqrt{\frac{T}{n}}\sigma_1 \xi_m^1\right), \qquad t_k \le t < t_{k+1}, \tag{4}$$

$$S_t^{2n} = \prod_{m=1}^{k} \left(1 + \sqrt{\frac{T}{n}}\sigma_2 \xi_m^2\right), \qquad t_k \le t < t_{k+1}. \tag{5}$$

In this model, the portfolio is revised only at times t_k, hence the pay-off process is

$$Y_t^n = F(t_k, S^n), \qquad t_k \le t < t_{k+1},$$

and the self-financing portfolios with initial capital x

$$V^{\pi^n,x} := x + \pi_-^n \cdot S^{1n},$$

where

$$\pi_t^n = \sum_{i=0}^{n-1} \pi_i^n \mathbb{I}_{[t_i,t_{i+1}[}(t), \qquad \pi_i^n \text{ is } \mathcal{F}_{t_i}\text{-measurable}.$$

Let

$$\mathcal{A}_x^n = \left\{\pi^n : V^{\pi^n,x} \ge 0\right\}.$$

We define the n-step shortfall risks by analogy with the previous section,

$$\begin{aligned} R^n(\pi^n, x) &= \sup_{\tau\in\mathcal{T}^n} E(Y_\tau^n - V_\tau^{\pi^n,x})^+, \\ R^n(x) &= \inf_{\pi^n\in\mathcal{A}_x^n} R^n(\pi^n, x), \end{aligned}$$

where $\mathcal{T}^n$ is the set of all stopping times with respect to $\mathbf{F^n}$.

The main result of the note is the following:

Theorem 3.1. *For any $x > 0$,*

$$R^n(x) \to R(x), \qquad n \to \infty.$$

The claim follows from the inequalities $\liminf R^n(x) \ge R(x)$ and $\limsup R^n(x) \le R(x)$. To establish the first one, we extract a convergent subsequence of almost optimal strategies for $R^n(x)$ and show that the limit is an admissible strategy for the initial model. We obtain the last inequality by approximating an almost optimal strategy for the initial model by n-step admissible strategies.

We shall develop several convergence results, on one hand, intrinsic to the traded asset price process (convergence of martingale measure), and on the other hand, related to the filtration (convergence of strategies).

4. Preliminary results

The approximating models give the convergence of laws $\mathscr{L}(S^n) \to \mathscr{L}(S)$ on $(\mathbb{D}(\mathbb{R}^2), \|\cdot\|_T)$, see Lemma 4.5 below. However, this result does not lead neither to option prices convergence nor to optimal strategy convergence. This section aims at studying more involved types of convergence.

We investigate the law convergence of càdlàg processes on the Skorohod space. We first recall the following results from [8]. Suppose that the limit law of a convergent sequence is the law of a continuous process. Then, the usual Skorohod distance coincides with the uniform distance on the Skorohod space. Moreover, if the sequence is vector–valued, the convergence on the space $(\mathbb{D}(\mathbb{R}^d), \|\cdot\|_T)$ (d is the dimension of the vector) is equivalent to the convergence of each coordinate separately on the space $(\mathbb{D}(\mathbb{R}), \|\cdot\|_T)$. To this end, we shall use this result without further mention.

Equivalent martingale measure

We now focus on the sequence of unique equivalent martingale measures given by the complete two assets model where the risky asset is driven by S^{n1}. We denote by Z^n the density process of these equivalent martingale measures. One can verify that

$$Z_T^n = \prod_{k=1}^{n} (1 + \Delta q_{t_k}^n) = \mathcal{E}(q^n)$$

where

$$\Delta q_{t_k}^n = -\frac{\mu^1}{\sigma^1}\sqrt{\frac{T}{n}}\xi_k^1.$$

For the continuous-time model, we consider the following equivalent martingale measure. Set

$$dq = -\frac{\mu^1}{\sigma^1} dW^1.$$

The martingale $Z = \mathcal{E}(q)$ is the density of the unique equivalent martingale measure of the so-called Black-Scholes model with one risky asset driven by S^1. Note that these martingales correspond to the densities of the minimal martingale measures in the incomplete markets. It can be checked with the explicit formulae given by Ansel and Stricker in [3]. The following lemma states the convergence in law of Z^n.

Lemma 4.1. *We have the convergence*

$$\mathscr{L}(Z^n) \to \mathscr{L}(Z)$$

on $(\mathbb{D}(\mathbb{R}), \|\cdot\|_T)$.

Proof. The Donsker theorem, see the discussion on the uniform topology in $\mathbb{D}(\mathbb{R})$ in [4], implies that

$$\mathscr{L}\left(\sum_{t_k=0}^{\cdot}\sqrt{\frac{T}{n}}\xi_k\right) \to \mathscr{L}(W^1).$$

By virtue of Cor 6.VI.29 in [8], we have convergence of the quadratic variation processes, namely

$$\mathscr{L}\left(\sum_{t_k=0}^{\cdot}\sqrt{\frac{T}{n}}\xi_k, \left[\sum_{t_k=0}^{\cdot}\sqrt{\frac{T}{n}}\xi_k\right]\right) \to \mathscr{L}(W^1, [W^1]).$$

Set $\Phi(x) = \ln(1+x) - x + \frac{x^2}{2}$. Note that

$$\Phi(x) = O(x^3), \qquad x \to 0.$$

Since $\|\Delta q^n\|_T = O(n^{-1/2})$, we get the asymptotic

$$\sum \Phi(\Delta q^n) = O(n^{-1/2}).$$

Then we deduce the convergence of the laws

$$\mathscr{L}(q^n, [q^n], \sum \Phi(\Delta q^n)) \to \mathscr{L}(q, [q], 0).$$

We refer to the following Lemma 4.2 to ensure the convergence of stochastic exponential. The proof is achieved. □

Lemma 4.2. *Let X^n, X be scalar adapted processes where X is continuous and such that*

$$\mathscr{L}\left(X^n, [X^n], \sum \Phi(\Delta X^n)\right) \to \mathscr{L}\left(X, [X], 0\right),$$

with $\Phi(x) = \ln(1+x) - x + \frac{x^2}{2}$. Then we have the following convergence in law of stochastic exponentials:

$$\mathscr{L}(\mathcal{E}(X^n)) \to \mathscr{L}(\mathcal{E}(X)).$$

Proof. The claim follows from

$$\mathcal{E}(X) = G\left(X, [X], \sum \Phi(\Delta X)\right),$$

with

$$G(x, y, z) = \exp\left(x - \frac{y}{2} + z\right).$$

Since G is continuous on $(\mathbb{D}(\mathbb{R}^3), \|.\|_T)$, we get the result. □

The next lemma provides explicit formulae for the consistent price processes Z^nS^{n1} and ZS^1.

Lemma 4.3. *The martingale ZS^1 is an Itô process. There exists a positive sequence $\tilde{\sigma}^{n1} \in \mathbb{R}_+$ such that $Z^nS^{n1} = \mathcal{E}(X^{n1})$ where*

$$\Delta X^{n1}_{t_k} = \tilde{\sigma}^{n1}\sqrt{\frac{T}{n}}\xi^1_k.$$

Analogically, there exists $\tilde{\sigma}^1 > 0$ such that $ZS^1 = \mathcal{E}(X^1)$ where

$$dX^1 = \tilde{\sigma}^1 dW^1.$$

Moreover $\tilde{\sigma}^{n1} \to \tilde{\sigma}^1$.

Proof. Note that

$$\ln(Z^n S^{n1})_{t_k} = \sum_{l=0}^{k} \ln\left(\left(1 + \mu^1 \frac{T}{n} + \sigma^1 \sqrt{\frac{T}{n}} \xi_l^1\right)\left(1 - \frac{\mu^1}{\sigma^1}\sqrt{\frac{T}{n}}\xi_l^1\right)\right).$$

After a direct computation, we get

$$\ln(Z^n S^{n1})_{t_k} = \sum_{l=0}^{k} \ln\left(1 + \sigma^1 \sqrt{\frac{T}{n}} \xi_l^1 1 - \frac{\mu^1}{\sigma^1}\sqrt{\frac{T}{n}}\xi_l^1 - \frac{(\mu^1)^2}{\sigma^1}\frac{T}{n}\sqrt{\frac{T}{n}}\xi_l^1\right).$$

We obtain $Z^n S^{n1}$ by taking the exponential. We identify

$$\tilde{\sigma}^{n1} = \sigma^1 - \frac{\mu^1}{\sigma^1} - \frac{(\mu^1)^2}{\sigma^1}\frac{T}{n}.$$

Let us compute ZS. We have

$$\begin{aligned} Z_t S_t^1 &= \exp\left(\sigma^1 W_t^1 + \left(\mu^1 - \frac{1}{2}(\sigma^1)^2\right)t - \frac{\mu^1}{\sigma^1}W_t^1 + \frac{1}{2}\left(\frac{\mu^1}{\sigma^1}\right)^2 t\right) \\ &= \exp\left(\left(\sigma^1 - \frac{\mu^1}{\sigma^1}\right)W_t^1 - \frac{1}{2}\left(\sigma^1 - \frac{\mu^1}{\sigma^1}\right)^2 t\right). \end{aligned}$$

With $\tilde{\sigma}^1 = \sigma^1 - \mu^1/\sigma^1$, we get the result. □

Remark 4.4. The processes X^n and X are the so-called stochastic logarithm of $Z^n S^{n1}$ and ZS^1. Moreover, the piecewise constant processes X^{n1} jumps only at dates t_k, $k \geq 1$. Namely, we have:

$$\Delta X_{t_k}^{n1} = (Z^n S^{n1})_{t_{k-1}}^{-1} \Delta (Z^n S^{n1})_{t_k} = (Z^n S^{n1})_{t_{k-1}}^{-1}((Z^n S^{n1})_{t_k} - (Z^n S^{n1})_{t_{k-1}}).$$

We get also

$$X^1 := ((ZS^1)_-)^{-1} \cdot ZS^1.$$

Finally, we state the joint convergence of the asset price processes together with the consistent price systems.

Lemma 4.5. *The following convergence holds*

$$\mathscr{L}(S^n, Z^n, Z^n S^{n1}) \to \mathscr{L}(S, Z, ZS^1).$$

Proof. We use the same type of arguments than for the proof of Lemma 4.1. The stochastic logarithms are given by formulae (1), (2), (4) and (5). The proof is then straightforward by means of Lemma 4.3. □

UT condition

The uniform tightness condition, UT condition in short, mainly stands to ensure convergence of stochastic integrals. Usually, this condition is defined for processes with an infinite time horizon. We can verify that our definition is consistent with the original one.

Let $\mathcal{H}^n$ be the set of all simple $\mathbf{F}^n$-predictable processes bounded by 1, i.e. of the form

$$H_s^n = h_0^n + \sum_{i=0}^{k} h_i^n \mathbb{I}_{]s_i,s_{i+1}]}(s)$$

where h_i^n is $\mathcal{F}_{s_i}^n$-measurable random variable with $|h_i^n| \leq 1$ and $\{s_0, \cdots, s_k\}$ is a partition of $[0,T]$. We say that S^n satisfies *UT condition* if the set $\{H^n \cdot S_T^n, H^n \in \mathcal{H}^n, n \in \mathbb{N}\}$ is bounded in probability.

Lemma 4.6. *The sequences of processes (S^{n1}), (S^{n2}) and $(Z^n S^{n1})$ verify UT condition.*

Proof. We shall prove the result for the sequence (S^{n1}). With analogous argument, we can extend the result to (S^{n2}) and $(Z^n S^{n2})$, writing σ_2 or $\tilde{\sigma}^1$ instead of σ_1 and $\mu_1 = 0$. Consider the following canonical decompositions

$$S^{1n} = M^{1n} + B^{1n},$$

where the processes B^{n1} and M^{n1} are piecewise constant with the jumps

$$\begin{aligned} \Delta M_{t_k}^{1n} &= \sqrt{\frac{T}{n}}\sigma_1 \xi_k^1 S_{t_{k-1}}^{1n}, \qquad \Delta\langle M^{1n}\rangle_{t_k} = \frac{T}{n}\sigma_1^2 (S_{t_{k-1}}^{1n})^2, \\ \Delta B_{t_k}^{1n} &= \frac{T}{n}\mu_1 S_{t_{k-1}}^{1n}. \end{aligned}$$

Let $H^n \in \mathcal{H}^n$. Then

$$||H^n \cdot B^{1n}||_T \leq \sum_{k=0}^{n} |\triangle B_{t_k}^{1n}| \leq \mu_1 T ||S^{1n}||_T.$$

The sequence (S^{1n}) is tight since the sequence of law is convergent. Hence $(||H^n \cdot B^{1n}||_T)$ is bounded in probability. Let us show that $(H^n \cdot M_t^{1n})$ is bounded in probability for any $t \in [0,T]$. By the Chebyshev inequality and Ito isometry, we have for any $K > 0$

$$P(|H^n \cdot M_t^{1n}| > K) \leq \frac{1}{K^2} E|H^n \cdot M^{1n}|_t^2 \leq \frac{1}{K^2} E\langle M^{1n}\rangle_t \leq \sum_{k=0}^{n} \frac{1}{K^2}\frac{\sigma_1^2 T}{n} E(S_{t_k}^{1n})^2.$$

It remains to observe that

$$\begin{aligned} E(S_{t_k}^{1n})^2 &= \prod_{m=1}^{k} E\left(1 + \mu_1\frac{T}{n} + \sqrt{\frac{T}{n}}\sigma_1\xi_m^1\right)^2 \\ &= \prod_{m=1}^{k}\left(1 + \frac{T}{n}\left(\mu_1^2\frac{T}{n} + \sigma_1^2 + 2\mu_1\right)\right) \leq C. \end{aligned}$$

The result follows. □

Remark 4.7. Note that this last observation is similar for (S^{2n}). Uniform integrability for the family $(S^n_t)_{0\le t\le T, n\in\mathbb{N}}$ follows. It is also possible to deduce uniform integrability for the sequence of random variables $(||S^n||_T)_{n\in\mathbb{N}}$. Since (S^n) is tight, for any $\varepsilon>0$, there exists a positive constant κ such that $\sup_n P(||S^n||_T>\kappa)\le\varepsilon$, we have

$$E||S^n||_T\mathbb{I}_{\{||S^n||_T>\kappa\}}=\sum_{t\le T}E|S^n_t|\mathbb{I}_{\{|S^n_t|=||S^n||_T\}}\mathbb{I}_{\{||S^n||_T>\kappa\}}\le\varepsilon E|S^n_T|^2.$$

Convergence of filtrations and extended convergence

To ensure convergence of strategies or stopping times, we need convergence of conditional expectations, which will hold with convergence of filtrations and extended convergence. We remind some basic results about these types of convergence following the survey of Coquet, Mémin and Slominski [5]. By virtue of Skorohod Representation Theorem, we define a probability space on which some subsequences of S^n converge to S almost surely under the Skorohod topology.

Definition 4.8. A sequence of filtrations $\mathbf{F^n}$ converges weakly to the filtration $\mathbf{F}$ if and only if for any $B\in\mathcal{F}_T$, the sequence of càdlàg martingales $(E\mathbb{I}_B|\mathcal{F}^n_\cdot)$ converges in probability to the martingale $(E\mathbb{I}_B|\mathcal{F}_\cdot)$ on $(\mathbb{D}(\mathbb{R}),\|\cdot\|_T)$.

Lemma 4.9. *Along the subsequences of (S^n) which converge almost surely to S, the filtrations $\mathbf{F^n}$ converge weakly to the filtration $\mathbf{F}$.*

Proof. First, we remark that if $S^n\xrightarrow{P}S$ on $(\mathbb{D}(\mathbb{R}^2),\|\cdot\|_T)$, we have also $\ln S^n\xrightarrow{P}\ln S$ on $(\mathbb{D}(\mathbb{R}^2),\|\cdot\|_T)$, where $\ln X=(\ln X^1,\ln X^2)$. For any $n\in\mathbb{N}$, the process $\ln S^n$ has independent increments. As the filtrations $\mathbf{F^n}$ (resp. $\mathbf{F}$) are also the natural filtrations of $\ln S^n$ (resp. $\ln S$), according to Proposition 2 in [5], $\mathbf{F^n}$ converges weakly to $\mathbf{F}$. □

The following was introduced as a characterization of extended convergence by Aldous in [1]. We shall use it as a definition.

Definition 4.10. Let us consider càdlàg processes X^n, X taking values in $\mathbb{R}^d$ and their natural filtrations $\mathbf{F^n},\mathbf{F}$. Then $X^n\Rrightarrow X$ holds if and only if for any integer k and for any real valued bounded functions $\psi_1,\cdots,\psi_k$, continuous on $\mathbb{D}(\mathbb{R}^d)$ endowed with the Skorohod distance d_S, we have

$$\mathscr{L}(X^n,E[\psi_1(X^n)|\mathcal{F}^n_\cdot],\cdots,E[\psi_k(X^n)|\mathcal{F}^n_\cdot])\to\mathscr{L}(X,E[\psi_1(X)|\mathcal{F}_\cdot],\cdots,E[\psi_k(X)|\mathcal{F}_\cdot])$$

on $(\mathbb{D}(\mathbb{R}^{d+k}),d_S)$.

Lemma 4.11. *Along the subsequences of (S^n) which converge almost surely to S,*

$$S^n\Rrightarrow S$$

in probability.

Proof. The sequence of filtrations $\mathbf{F^n}$ converges weakly to $\mathbf{F}$. As S is continuous, the result is immediate with remark 1 in [5]. $\square$

Convergence of "almost optimal" portfolios

We end this section with a preliminary result concerning convergence of portfolios. The following Lemma is the key step to prove the inequality $R(x) \leq \liminf R^n(x)$. We point out the convergence of a sequence of almost optimal strategies to an admissible strategy in the model driven by S.

Lemma 4.12. *For any $x > 0$, for any $\varepsilon > 0$, there exists a sequence $(\pi^n)_{n\in\mathbb{N}}$, $\pi^n \in \mathcal{A}^n_x$,*

$$R^n(\pi^n) < R^n(x) + 1/n + \varepsilon$$

such that $\mathscr{L}(S^n, Z^n, Z^n V^{\pi^n,x}) \to \mathscr{L}(S, Z, M)$ on the space $(\mathbb{D}(\mathbb{R}^4), \|\cdot\|)$. Moreover, for the càdlàg process M, there exists an admissible strategy $\pi \in \mathcal{A}^x$ such that $M = ZV^{\pi,x}$.

Proof. Let $x > 0$, for any n, there exists $\gamma^n \in \mathcal{A}^n_x$ such that

$$R^n(\gamma^n) < R^n(x) + \frac{1}{n}.$$

Choose $\varepsilon > 0$. Since $(||S^n||_T)$ is uniformly integrable, there exists a constant $\kappa > 0$ such that

$$\sup_{n\in\mathbb{N}} E||S^n||_T \mathbb{I}_{\{||S^n||_T \geq \kappa/C\}} \leq \varepsilon/C,$$

C is the constant defined by (3). Define the stopping times

$$\tau_n = \min\{t : V^{\gamma^n,x}_t \geq \kappa\},$$

and set π^n the admissible strategy

$$\pi^n = \gamma^n \mathbb{I}_{[0,\tau_n[}.$$

It follows that, for any n, any $\tau \in \mathcal{T}^n$,

$$\begin{aligned}
E(Y^n_\tau - V^{\pi^n,x}_\tau)^+ &\leq E(Y^n_\tau - V^{\gamma^n,x}_\tau)^+ + E(Y^n_\tau - V^{\pi^n,x}_\tau)^+ \mathbb{I}_{\{Y^n_\tau > \kappa\}} \mathbb{I}_{\{V^{\gamma^n,x} > \kappa\}} \\
&\leq E(Y^n_\tau - V^{\gamma^n,x}_\tau)^+ + CE||S^n||_T \mathbb{I}_{\{||S^n||_T \geq \kappa/C\}} \mathbb{I}_{\{V^{\gamma^n,x} > \kappa\}} \\
&\leq E(Y^n_\tau - V^{\gamma^n,x}_\tau)^+ + \varepsilon.
\end{aligned}$$

Hence, for any n, π^n satisfies

$$R^n(\pi^n) < R^n(x) + 1/n + \varepsilon.$$

Note that we have constructed a sequence of portfolios such that $\sup_n \|V^{\pi^n,x}\|_T$ is bounded. Indeed, we have

$$V^{\pi^n,x}_{t_{k+1}} = V^{\pi^n,x}_{t_k} + \pi^n_{t_k} \bigtriangleup S^{1n}_{t_{k+1}},$$

for any k, any n. Since $V^{\pi^n,x}_{t_k} \leq \kappa$, the admissibility condition implies

$$0 \leq \kappa + \pi^n_{t_k} \bigtriangleup S^{1n}_{t_{k+1}}.$$

It follows that

$$0 \le \kappa + \pi^n_{t_k} S^{1n}_{t_k}\left(\mu^1 \frac{T}{n} + \sigma^1\sqrt{\frac{T}{n}}\xi^1_{k+1}\right).$$

Recall that $\pi^n_{t_k}$ is $\mathcal{F}^n_{t_k}$-measurable. When $\xi^1_{k+1} = -1$, we get

$$\pi^n_{t_k} \le \sqrt{\frac{n}{T}}\frac{\kappa}{S^{1n}_{t_k}\left(\sigma^1 - \mu^1\sqrt{T/n}\right)}.$$

When $\xi^1_{k+1} = 1$, we have

$$\pi^n_{t_k} \ge -\sqrt{\frac{n}{T}}\frac{\kappa}{S^{1n}_{t_k}\left(\sigma^1 + \mu^1\sqrt{T/n}\right)}.$$

This leads to the inequality

$$|\pi^n_{t_k}| \le \sqrt{n}\ \frac{C}{S^{1n}_{t_k}},$$

for n large enough. Hence

$$|\pi^n_{t_k} \triangle S^{1n}_{t_{k+1}}| \le C_1\sqrt{n}\left(\mu_1\frac{T}{n} + \sigma_1\sqrt{\frac{T}{n}}\right) \le C_2.$$

This inequality remains true for stopping times, proving that $\sup_n \|V^{\pi^n}\|_T$ is bounded. We remark that we can deduce the following bounds for the strategy π^n:

$$|\pi^n_{t_k} S^{n1}_{t_k}| \le \frac{C_1 S^{n1}_{t_k}}{|\triangle S^{n1}_{t_{k+1}}|} \le C. \tag{6}$$

Let us investigate the convergence of the càdlàg $\mathbf{F^n}$-martingales $M^n = Z^n V^{\pi^n}$. We shall apply Th. VI.4.13 and Prop. VI.3.26 of [8] to prove that the sequence M^n is C-tight (that is M^n is tight and each cluster point is the law of a continuous process). The sequence of initial values $M^n_0 = x$ is bounded. We need to prove that the sequence of processes $\langle M^n\rangle$ is C-tight. First, the sequence of processes $\langle M^n\rangle$ is uniformly bounded in probability. Indeed, by the Itô Isometry and Remark 4.3, we have

$$E|\langle M^n\rangle_T|^2 = E\int_0^T (\pi^n_u)^2 d[Z^n S^n]_u = E\sum_{k=0}^{n-1}(\pi^n_{t_k} Z^n_{t_k} S^{n1}_{t_k})^2(\tilde{\sigma}^1)^2 T/n.$$

The inequality (6) implies that

$$E|\langle M^n\rangle_T|^2 \le C.$$

This ensures the boundedness in probability.

For a function $\alpha \in \mathbb{D}(\mathbb{R})$ we define the modulus of continuity $w(\alpha,\delta)$, $\delta > 0$, by the formula

$$w(\alpha,\delta) := \sup\{|\alpha_{t+h} - \alpha_t| :\ t \in [0, T-\delta],\ h \in [0,\delta]\}.$$

The next step to study the tightness of $\langle M^n\rangle$ is to show that its modulus of continuity is arbitrarily small when $\delta \to 0$. Note that

$$\langle M^n\rangle_{t_{k+l}} - \langle M^n\rangle_{t_k} = \sum_{i=0}^{l-1} (\pi^n_{t_{k+i}})^2 \Delta\langle Z^n S^{n1}\rangle_{t_{k+i+1}}. \tag{7}$$

We deduce from Burkholder–Davis–Gundy inequality, Remark 4.3 and (6) that

$$E \sup_{k\le n-l} \left|\langle M^n\rangle_{t_{k+l}} - \langle M^n\rangle_{t_k}\right|^2 \le E\sum_{i=0}^{l-1} (\pi^n_{t_{k+i}} Z^n_{t_{k+i}} S^{n1}_{t_{k+i}})^4 (\tilde\sigma^1)^4 (T/n)^2 \le \kappa (l/n)^2.$$

We deduce that there is a constant $C>0$ such that for any $\delta>0$ we have, for all sufficiently large n, the inequality

$$E\left|w(\langle M^{ni}\rangle,\delta)\right|^2 \le C(\delta + T/n)^2.$$

By virtue of Th. 15.5 in [4] the sequence $\langle M^n\rangle$ is C-tight. Th. VI.4.13 and Prop. VI.3.26 of [8] give the C-tightness of M^n.

Consider a cluster point $\mathscr{L}(M)$ of the sequence $\mathscr{L}(M^n)$. We show that it follows the same law as $ZV^{\pi,x}$, where $\pi\in\mathcal{A}^x$. Set $X^{n2} = (S^{n2}_-)^{-1}\cdot S^{n2}$. Note that, thanks to characterization (7), there exists a sequence $(m^n)_{n\in\mathbb{N}}$ of non-negative uniformly bounded predictable processes such that

$$\Delta\langle (M^n, X^{n2}/\tilde\sigma^2)\rangle_{t_k} = \begin{pmatrix} m^n_{t_k} & 0\\ 0 & 1\end{pmatrix}\frac{T}{n}.$$

The martingale $(M^n, X^{n2}/\tilde\sigma^2)$ converges in law to the process (M,W^2) (maybe along a subsequence). Since we have the convergence of the sequence of quadratic characteristics, we deduce that

$$d\langle (M,W^2)\rangle_t = \begin{pmatrix} m & 0\\ 0 & t\end{pmatrix} dt,$$

with m a non-negative uniformly bounded predictable process. According to [9], Th. 3.4.2, there is a filtered probability space $(\Omega,\mathcal{F},\mathbf{F},R)$ with a standard Brownian motion $B=(B^1,B^2)$ and the matrix-valued process g such that R-a.s. we have

$$\begin{aligned} M &= g^{11}\cdot B^1 + g^{12}\cdot B^2,\\ W^2 &= g^{21}\cdot B^1 + g^{22}\cdot B^2. \end{aligned}$$

with

$$\langle M, W^2\rangle_t = \int_0^t gg'_s ds.$$

Note that the process $((g^{11}\cdot B^1 + g^{12}\cdot B^2)/\sqrt{m}, W^2)$ is a standard Brownian motion. Finally, there exists a process m such that M and $x+\sqrt{m}\cdot W^1$ have the same law. According to Remark 4.4, we get

$$\begin{aligned} x+\sqrt{m}\cdot W^1 &= \frac{\sqrt{m}}{\tilde{\sigma}^1}\cdot X^1 \\ &= \frac{\sqrt{m}}{\tilde{\sigma}^1}(ZS^1)_-\cdot ZS^1 \\ &= Z\left(\frac{\sqrt{m}}{\tilde{\sigma}^1}(ZS^1)_-\cdot S^1\right). \end{aligned}$$

Set the predictable process

$$\pi=\frac{\sqrt{m}}{\tilde{\sigma}^1}(ZS^1)_-.$$

It remains to check that π is an admissible strategy to get the result. Fix $t\in[0,T]$. Since Z^n, Z are strictly positive martingales, we have

$$P(Z^nV_t^{\pi^n,x}\geq 0)=1.$$

According to Portmanteau Theorem, see [4], since $\mathscr{L}(Z^nV_t^{\pi^n,x})\to\mathscr{L}(ZV_t^{\pi,x})$, we have

$$\limsup P(Z^nV_t^{\pi^n,x}\geq 0)\leq P(ZV^{\pi,x}\geq 0).$$

It follows that $V^\pi\geq 0$ almost everywhere. The proof is achieved. □

5. Proof of the inequality $R(x)\leq\liminf R^n(x)$

Choose $x>0$, $\varepsilon>0$. We implicitly consider a subsequence such that $R^n(x)\to\liminf R^n(x)$. According to Lemma 4.12, for any n, there exists a sequence $(\pi^n)_{n\in\mathbb{N}}$, $\pi^n\in\mathcal{A}_x^n$,

$$R^n(\pi^n)<R^n(x)+1/n+\varepsilon/3$$

and a probability space such that $(Y^n,Z^n,Z^nV^{\pi^n,x})\overset{a.s.}{\to}(Y,Z,ZV^{\pi,x})$ on the space $(\mathbb{D}(\mathbb{R}^3),\|\cdot\|_T)$, for some admissible strategy π. There exists $\tau_\varepsilon\in\mathcal{T}$ such that

$$R(\pi)<\frac{\varepsilon}{3}+E(Y_{\tau_\varepsilon}-V_{\tau_\varepsilon}^{\pi,x})^+.$$

For any k, there exists a finite set $I_k\subset[0,T]$, $T\in I_k$, such that $[0,T]$ is a subset of $\bigcup_{t\in I_k}]t-1/k,t+1/k[$. Set

$$\tau_k=\min\{t\in I_k, t\geq\tau_\varepsilon\}.$$

Note that $\tau_k\in\mathcal{T}$ and $|\tau_k-\tau|<2/k$. Hence $\tau_k\searrow\tau$ a.s. when $k\to\infty$. The process $\zeta_t=(Y_t-V_t^{\pi,x})^+$ is uniformly integrable because of assumption (3) on F. It follows that, choosing k large enough, there exists a stopping time $\tau:=\tau_k$ in $\mathcal{T}$ admitting only a finite number of values $\{t_1,\cdots,t_m\}\subset I_k$ such that

$$R(\pi)<\frac{2\varepsilon}{3}+E(Y_\tau-V_\tau^{\pi,x})^+.$$

Set $\tau_n(\omega) = \min\{t_i \in I_k : E[\mathbb{I}_{\{\tau=t_i\}}|\mathcal{F}^n_{t_i}] > 1/2\}$. Observe that

$$\{\tau_n \neq \tau\} \subset \bigcup_{i=1}^{m} \left\{|E[\mathbb{I}_{\{\tau=t_i\}}|\mathcal{F}^n_{t_i}] - \mathbb{I}_{\{\tau=t_i\}}| \geq 1/2\right\}.$$

Because of convergence of filtrations and Lemma 4.9, we have $P\{\tau_n \neq \tau\} \to 0$. Hence, the sequence converges to τ almost surely. Furthermore $\tau_n \in \mathcal{T}^n$ takes values in $\{t_1, \cdots, t_m\}$, as well as τ. Recalling that

$$\begin{aligned} R(x) \leq R(\pi) &< \frac{2\varepsilon}{3} + E(Y_\tau - V^{\pi,x}_\tau)^+ \\ &\leq \frac{2\varepsilon}{3} + E\left(Y_\tau - \frac{(ZV^{\pi,x})_\tau}{Z_\tau}\right)^+, \end{aligned}$$

we deduce that

$$R(x) \leq \frac{2\varepsilon}{3} + E \lim_{a.s.} \left(Y^n_{\tau_n} - \frac{Z^n_{\tau_n} V^{\pi^n,x}_{\tau_n}}{Z^n_{\tau_n}}\right)^+.$$

From Fatou's Lemma, we get

$$\begin{aligned} R(x) &\leq \frac{2\varepsilon}{3} + \liminf_{n\to\infty} E\left(Y^n_{\tau_n} - V^{\pi^n,x}_{\tau_n}\right)^+ \\ &\leq \frac{2\varepsilon}{3} + \liminf_{n\to\infty} R^n(\pi^n) \leq \varepsilon + \liminf_{n\to\infty} R^n(x). \end{aligned}$$

As ε is arbitrary, it follows that $R(x) \leq \liminf R^n(x)$.

6. Proof of the inequality $R(x) \geq \limsup R^n(x)$

Choose $x > 0$. The sequence is implicitly reindexed such that the whole sequence tends to $\limsup R^n(x)$. By virtue of Skorohod Representation Theorem, one can suppose that S^n, S are defined on a same probability space such that

$$S^n \xrightarrow{P} S, \qquad \text{on } (\mathbb{D}(\mathbb{R}), \|\cdot\|_T).$$

Choose $\varepsilon > 0$. There exists π, an admissible strategy such that

$$R(x) + \varepsilon > \sup_{\tau\in\mathcal{T}} E(Y_\tau - V^{\pi,x}_\tau)^+.$$

The process $V^{\pi,x}$ is continuous. We can build some n-step portfolios converging to $V^{\pi,x}$. More precisely, we approximate the strategy π. Fix $\varepsilon > 0$. There exist i_0 and Borelian functions g_i on $\mathbb{D}(\mathbb{R})$ such that:

$$E\int_0^T \left|\pi_s - \sum_0^{i_0} g_i(S^{s_i})\mathbb{I}_{[s_i,s_{i+1}[}(s)\right|^2 \mathrm{d}s \leq \varepsilon,$$

see Th. 4.41 in [2]. The notation S^{s_i} stands for the stopped process, i.e. $S^{s_i} = S\mathbb{I}_{[0,s_i]} + S_{s_i}\mathbb{I}_{]s_i,T]}$. According to Th. 4.33 in [2], each g_i is (everywhere) the pointwise limit of bounded continuous functions on $\mathbb{D}(\mathbb{R})$ (endowed with Skorohod distance). It follows that there exists some bounded continuous functions ψ_i such that,

$$E\int_0^T \left|\pi_s - \sum \psi_i(S^{s_i})\mathbb{I}_{[s_i,s_{i+1}[}(s)\right|^2 \mathrm{d}s \leq 2\varepsilon.$$

Set the strategy

$$\tilde{\pi}_t = \sum \psi_i(S^{s_i})\mathbb{I}_{[s_i,s_{i+1}[}(t).$$

According to Burkholder–Davis–Gundy inequality, observe that

$$E\left\|\pi \cdot S^1 - \tilde{\pi} \cdot S^1\right\|_T \leq C\varepsilon.$$

Then, we have

$$E\|V^{\pi,x} - V^{\tilde{\pi},x}\|_T \leq C\varepsilon. \tag{8}$$

Define

$$t^n(s) = max\{t_k^n : t_k^n \leq s\}, \qquad i(s) = max\{i : s_i \leq s\}.$$

Set for large n

$$\pi_t^n = E\left[\psi_{i(t)}\left((S^n)^{t^n(s_{i(t)})}\right)\middle|\mathcal{F}_t^n\right].$$

By virtue of extended convergence (Lemma 4.11),

$$(S^n, \pi^n) \xrightarrow{P} (S, \tilde{\pi})$$

on $(\mathbb{D}(\mathbb{R}^3), \|\cdot\|_T)$. According to convergence of stochastic integrals, Th. VI.6.22 in [8], involving UT Condition for S^{n1}, we get

$$(S^n, V^{\pi^n,x}) \xrightarrow{P} (S, V^{\tilde{\pi},x})$$

on $(\mathbb{D}(\mathbb{R}^3), \|\cdot\|_T)$. For any n, there exists a stopping time τ_n satisfying

$$R^n(\pi^n) - 1/n \leq E(Y_{\tau_n}^n - V_{\tau_n}^{\pi^n,x})^+.$$

The sequence (S^n, V^{π^n}, τ_n) is tight on the space $(\mathbb{D}(\mathbb{R}^3), \|\cdot\|_T) \times [0,T]$. Thus, there exists a subsequence, a random variable $\nu \leq T$ and a probability space given by Skorohod Representation Theorem, such that

$$(Y^n, V^{\pi^n,x}, \tau_n) \xrightarrow{P} (Y, V^{\tilde{\pi},x}, \nu)$$

on the space $(\mathbb{D}(\mathbb{R}^2), \|\cdot\|_T) \times [0,T]$. There exists some sequences $\delta_n, \varepsilon_n \searrow 0$,

$$A_n = \{\|Y^n - Y\|_T + ||V^{\pi^n} - V^{\tilde{\pi},x}||_T \leq \delta_n\}, \quad E||S^n||_T\mathbb{I}_{A_n^c} \leq \varepsilon_n.$$

The following inequalities hold for all $n \in \mathbb{N}$. In one hand

$$\begin{aligned}
(Y^n - V^{\pi^n,x})^+\mathbb{I}_{A_n} &= (Y - V^{\tilde{\pi},x} + V^{\tilde{\pi},x} - V^{\pi^n,x} + Y^n - Y)^+\mathbb{I}_{A_n} \\
&\leq (Y - V^{\tilde{\pi},x} + |V^{\tilde{\pi},x} - V^{\pi^n,x}| + |Y^n - Y|)^+\mathbb{I}_{A_n} \\
&\leq (Y - V^{\tilde{\pi},x})^+\mathbb{I}_{A_n} + |V^{\tilde{\pi},x} - V^{\pi^n,x}|\mathbb{I}_{A_n} + |Y^n - Y|\mathbb{I}_{A_n} \\
&\leq (Y - V^{\tilde{\pi},x})^+\mathbb{I}_{A_n} + \delta_n;
\end{aligned}$$

on the other hand, by virtue of (3),

$$(Y^n - V^{\pi^n,x})^+ \mathbb{I}_{A_n^c} \le c||S^n||_T \mathbb{I}_{A_n^c}.$$

Then we have

$$(Y^n - V^{\pi^n,x})^+ \le (Y - V^{\tilde{\pi},x})^+ + \delta_n + \varepsilon_n. \tag{9}$$

We claim that for the uniformly integrable càdlàg process defined by $\Phi_t = (Y_t - V_t^{\tilde{\pi},x})^+$,

$$E\Phi_\nu \le \sup_{\tau \in \mathcal{T}} E\Phi_\tau. \tag{10}$$

The proof could be found in [1], chapter 5, or more concisely in Lemma 3.3 in [6]. It is now possible to obtain the following inequalities

$$\begin{aligned} \limsup_{n\to\infty} R^n(x) &\le \limsup_{n\to\infty} R^n(\pi^n) \\ &\le \limsup_{n\to\infty} E(Y^n_{\tau_n} - V^{\pi^n,x}_{\tau^n})^+. \end{aligned}$$

With help of reverse Fatou's Lemma, we have

$$\limsup_{n\to\infty} R^n(x) \le E \limsup_{n\to\infty} (Y^n_{\tau_n} - V^{\pi^n,x}_{\tau^n})^+.$$

According to inequality (9), we get

$$\limsup_{n\to\infty} R^n(x) \le E \limsup_{n\to\infty} (Y_{\tau_n} - V^{\tilde{\pi},x}_{\tau^n})^+ + \limsup_{n\to\infty} (\delta_n + C\varepsilon_n).$$

Finally, with (10) and (8), we obtain

$$\limsup_{n\to\infty} R^n(x) \le \sup_{\tau\in\mathcal{T}} E(Y_\tau - V_\tau^{\pi,x})^+ + C\varepsilon \le R(x) + (C+1)\varepsilon.$$

Which achieve the proof since ε is arbitrary.

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